

Derivations and skew derivations of the Grassmann algebras

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Abstract

Surprisingly, skew derivations rather than ordinary derivations are more basic (important) object in study of the Grassmann algebras. Let $\Lambda_n = K[x_1, \dots, x_n]$ be the Grassmann algebra over a commutative ring K with $\frac{1}{2} \in K$, and δ be a skew K -derivation of Λ_n . It is proved that δ is a unique sum $\delta = \delta^{ev} + \delta^{od}$ of an even and odd skew derivation. Explicit formulae are given for δ^{ev} and δ^{od} via the elements $\delta(x_1), \dots, \delta(x_n)$. It is proved that the set of all even skew derivations of Λ_n coincides with the set of all the inner skew derivations. Similar results are proved for derivations of Λ_n . In particular, $\text{Der}_K(\Lambda_n)$ is a faithful but not simple $\text{Aut}_K(\Lambda_n)$ -module (where K is reduced and $n \geq 2$). All differential and skew differential ideals of Λ_n are found. It is proved that the set of generic normal elements of Λ_n that are not units forms a single $\text{Aut}_K(\Lambda_n)$ -orbit (namely, $\text{Aut}_K(\Lambda_n)x_1$) if n is even and two orbits (namely, $\text{Aut}_K(\Lambda_n)x_1$ and $\text{Aut}_K(\Lambda_n)(x_1 + x_2 \cdots x_n)$) if n is odd.

Key Words: The Grassmann algebra, derivation, skew derivation, group of automorphisms, normal element, orbit.

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1 Introduction

Throughout, ring means an associative ring with 1. Let K be an arbitrary ring (not necessarily commutative). The *Grassmann algebra* (the *exterior algebra*) $\Lambda_n = \Lambda_n(K) = K[x_1, \dots, x_n]$ is generated freely over K by elements x_1, \dots, x_n that satisfy the defining relations:

$$x_1^2 = \cdots = x_n^2 = 0 \text{ and } x_i x_j = -x_j x_i \text{ for all } i \neq j.$$

The Grassmann algebra $\Lambda_n = \bigoplus_{i \in \mathbb{N}} \Lambda_{n,i}$ is an \mathbb{N} -graded algebra ($\Lambda_{n,i} \Lambda_{n,j} \subseteq \Lambda_{n,i+j}$ for all $i, j \geq 0$) where $\Lambda_{n,i} := \bigoplus_{|\alpha|=i} Kx^\alpha$, $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and $|\alpha| := \alpha_1 + \cdots + \alpha_n$.

Derivations of the Grassmann algebras. Let $\text{Der}_K(\Lambda_n)$, $\text{Der}_K(\Lambda_n)^{ev}$, $\text{Der}_K(\Lambda_n)^{od}$ and $\text{IDer}_K(\Lambda_n)$ be the set of all, even, odd and inner derivations of $\Lambda_n(K)$ respectively. Note that $\text{IDer}_K(\Lambda_n) = \{\text{ad}(a) \mid a \in \Lambda_n\}$ where $\text{ad}(a)(x) := ax - xa$. Let Λ_n^{ev} and Λ_n^{od} be the set of even and odd elements of Λ_n . Let $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ be partial skew K -derivations of Λ_n ($\partial_i(x_j) = \delta_{ij}$, the Kronecker delta, and $\partial_i(a_j a_k) = \partial_i(a_j) a_k + (-1)^j a_j \partial_i(a_k)$ for all $a_i \in \Lambda_{n,i}$ and $a_j \in \Lambda_{n,j}$).

• (Theorem 2.1) *Suppose that K is a commutative ring with $\frac{1}{2} \in K$. Then*

1. $\text{Der}_K(\Lambda_n) = \text{Der}_K(\Lambda_n)^{ev} \oplus \text{Der}_K(\Lambda_n)^{od}$.
2. $\text{Der}_K(\Lambda_n)^{ev} = \bigoplus_{i=1}^n \Lambda_n^{od} \partial_i$.
3. $\text{Der}_K(\Lambda_n)^{od} = \text{IDer}_K(\Lambda_n)$.
4. $\text{Der}_K(\Lambda_n) / \text{IDer}_K(\Lambda_n) \simeq \text{Der}_K(\Lambda_n)^{ev}$.

So, each derivation $\delta \in \text{Der}_K(\Lambda_n)$ is a unique sum $\delta = \delta^{ev} + \delta^{od}$ of an even and odd derivation. When K is a field of characteristic $\neq 2$ this fact was proved by Djokovic, [4]. For an even n , let $\Lambda_n'^{od} := \Lambda_n^{od}$. For an odd n , let $\Lambda_n'^{od}$ be the K -submodule of Λ_n^{od} generated by all ‘monomials’ x^α but $\theta := x_1 \cdots x_n$, i.e. $\Lambda_n^{od} = \Lambda_n'^{od} \oplus K\theta$. The next result gives explicitly derivations δ^{ev} and δ^{od} via the elements $\delta(x_1), \dots, \delta(x_n)$.

• (Corollary 2.5) *Let K be a commutative ring with $\frac{1}{2} \in K$, δ be a K -derivation of $\Lambda_n(K)$, and, for each $i = 1, \dots, n$, $\delta(x_i) = u_i^{ev} + u_i^{od}$ for unique elements $u_i^{ev} \in \Lambda_n^{ev}$ and $u_i^{od} \in \Lambda_n^{od}$. Then*

1. $\delta^{ev} = \sum_{i=1}^n u_i^{od} \partial_i$, and
2. $\delta^{od} = -\frac{1}{2} \text{ad}(a)$ where the unique element $a \in \Lambda_n'^{od}$ is given by the formula

$$a = \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(u_{i+1}^{ev}) + \partial_1(u_1^{ev}).$$

The next results describes differential ideals of Λ_n (i.e. which are stable under all derivations).

• (Proposition 2.6) *Let K be a commutative ring with $\frac{1}{2} \in K$, $\mathcal{F}_n(K) := \{I : I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \mid I_i \text{ are ideals of } K\}$ be the set of n -flags of ideals of K , $\text{DI}(\Lambda_n)$ be the set of all differentiable ideals of $\Lambda_n(K)$. Then the map*

$$\mathcal{F}_n(K) \rightarrow \text{DI}(\Lambda_n), I \mapsto \hat{I} := \bigoplus_{i=0}^n \bigoplus_{|\alpha|=i} I_i x^\alpha,$$

is a bijection. In particular, \mathfrak{m}^i , $0 \leq i \leq n+1$, are differential ideals of Λ_n ; these are the only differential ideals of Λ_n if K is a field of characteristic $\neq 2$.

- (Theorem 2.10) *Let K be a reduced commutative ring with $\frac{1}{2} \in K$, $n \geq 1$. Then*

1. *$\text{Der}_K(\Lambda_n)$ is a faithful $\text{Aut}_K(\Lambda_n)$ -module iff $n \geq 2$.*
2. *The $\text{Aut}_K(\Lambda_n)$ -module $\text{Der}_K(\Lambda_n)$ is not simple.*

Skew derivations of the Grassmann algebras. Let $\text{SDer}_K(\Lambda_n)$, $\text{SDer}_K(\Lambda_n)^{ev}$, $\text{SDer}_K(\Lambda_n)^{od}$ and $\text{ISDer}_K(\Lambda_n)$ be the set of all, even, odd and inner skew derivations of $\Lambda_n(K)$ respectively. $\text{ISDer}_K(\Lambda_n) = \{\text{sad}(a) \mid a \in \Lambda_n\}$ and $\text{sad}(a)(a_i) := aa_i - (-1)^i a_i a$ ($a_i \in \Lambda_{n,i}$) is the inner skew derivation determined by the element a . For an odd n , let $\Lambda_n'^{ev} := \Lambda_n^{ev}$. For an even n , let $\Lambda_n'^{ev}$ be the K -submodule of Λ_n^{ev} generated by all ‘monomials’ x^α but $\theta := x_1 \cdots x_n$, i.e. $\Lambda_n^{ev} = \Lambda_n'^{ev} \oplus K\theta$.

- (Theorem 3.1) *Suppose that K is a commutative ring with $\frac{1}{2} \in K$. Then*

1. $\text{SDer}_K(\Lambda_n) = \text{SDer}_K(\Lambda_n)^{ev} \oplus \text{SDer}_K(\Lambda_n)^{od}$.
2. $\text{SDer}_K(\Lambda_n)^{od} = \bigoplus_{i=1}^n \Lambda_n'^{ev} \partial_i$.
3. $\text{SDer}_K(\Lambda_n)^{ev} = \text{ISDer}_K(\Lambda_n)$.
4. $\text{SDer}_K(\Lambda_n)/\text{ISDer}_K(\Lambda_n) \simeq \text{SDer}_K(\Lambda_n)^{od}$.

So, any skew K -derivation δ of Λ_n is a unique sum $\delta = \delta^{ev} + \delta^{od}$ of an even and odd skew derivation, and $\delta^{ev} := \frac{1}{2}\text{sad}(a)$ for a unique element $a \in \Lambda_n'^{ev}$. The next corollary describes explicitly the skew derivations δ^{ev} and δ^{od} .

- (Corollary 3.2) *Let K be a commutative ring with $\frac{1}{2} \in K$, δ be a skew K -derivation of $\Lambda_n(K)$, and, for each $i = 1, \dots, n$, $\delta(x_i) = u_i^{ev} + u_i^{od}$ for unique elements $u_i^{ev} \in \Lambda_n^{ev}$ and $u_i^{od} \in \Lambda_n^{od}$. Then*

1. $\delta^{od} = \sum_{i=1}^n u_i^{ev} \partial_i$, and
2. $\delta^{ev} = \frac{1}{2}\text{sad}(a)$ where the unique element $a \in \Lambda_n'^{ev}$ is given by the formula

$$a = \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(u_{i+1}^{od}) + \partial_1(u_1^{od}).$$

The action of $\text{Aut}_K(\Lambda_n)$ on the set of generic normal non-units of Λ_n .

Let \mathcal{N} be the set of all the normal elements of the Grassmann algebra $\Lambda_n = \Lambda_n(K)$, \mathcal{U} be the set of all units of Λ_n , and $G := \text{Aut}_K(\Lambda_n)$ be the group of K -automorphisms of Λ_n . Then $\mathcal{U} \subseteq \mathcal{N}$. The set \mathcal{N} is a disjoint union of its G -invariant subsets,

$$\mathcal{N} = \bigcup_{i=0}^n \mathcal{N}_i, \quad \mathcal{N}_i := \{a \in \mathcal{N} \mid a = a_i + \cdots, 0 \neq a_i \in \Lambda_{n,i}\}.$$

Clearly, $\mathcal{N}_0 = \mathcal{U}$. The next result shows that ‘generic’ normal non-unit elements of Λ_n (i.e. the set \mathcal{N}_1) form a single G -orbit if n is even, and two G -orbits if n is odd.

- (Theorem 4.3) *Let K be a field of characteristic $\neq 2$ and $\Lambda_n = \Lambda_n(K)$. Then*

1. $\mathcal{N}_1 = Gx_1$ if n is even.
2. $\mathcal{N}_1 = Gx_1 \cup G(x_1 + x_2 \cdots x_n)$ is the disjoint union of two orbits if n is odd.

The stabilizers of the elements x_1 and $x_1 + x_2 \cdots x_n$ are found (Lemma 4.4 and Lemma 4.6).

2 Derivations of the Grassmann rings

In this section, the results on derivations from the Introduction are proved. First, we recall some facts on Grassmann algebra (more details the reader can find in [3]).

The Grassmann algebra and its gradings. Let K be an *arbitrary* ring (not necessarily commutative). The *Grassmann algebra* (the *exterior algebra*) $\Lambda_n = \Lambda_n(K) = K[x_1, \dots, x_n]$ is generated freely over K by elements x_1, \dots, x_n that satisfy the defining relations:

$$x_1^2 = \cdots = x_n^2 = 0 \text{ and } x_i x_j = -x_j x_i \text{ for all } i \neq j.$$

Let \mathcal{B}_n be the set of all subsets of the set of indices $\{1, \dots, n\}$. We may identify the set \mathcal{B}_n with the direct product $\{0, 1\}^n$ of n copies of the two-element set $\{0, 1\}$ by the rule $\{i_1, \dots, i_k\} \mapsto (0, \dots, 1, \dots, 1, \dots, 0)$ where 1's are on i_1, \dots, i_k places and 0's elsewhere. So, the set $\{0, 1\}^n$ is the set of all the characteristic functions on the set $\{1, \dots, n\}$.

$$\Lambda_n = \bigoplus_{\alpha \in \mathcal{B}_n} Kx^\alpha = \bigoplus_{\alpha \in \mathcal{B}_n} x^\alpha K, \quad x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

where $\alpha = (\alpha_1, \dots, \alpha_n) \in \{0, 1\}^n = \mathcal{B}_n$. Note that the order in the product x^α is fixed. So, Λ_n is a free left and right K -module of rank 2^n . Note that $(x_i) := x_i \Lambda_n = \Lambda_n x_i$ is an ideal of Λ_n . Each element $a \in \Lambda_n$ is a unique sum $a = \sum a_\alpha x^\alpha$, $a_\alpha \in K$. One can view each element a of Λ_n as a 'function' $a = a(x_1, \dots, x_n)$ in the non-commutative variables x_i . The K -algebra epimorphism

$$\begin{aligned} \Lambda_n &\rightarrow \Lambda_n / (x_{i_1}, \dots, x_{i_k}) \simeq K[x_1, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_k}}, \dots, x_n], \\ a &\mapsto a|_{x_{i_1}=0, \dots, x_{i_k}=0} := a + (x_{i_1}, \dots, x_{i_k}), \end{aligned}$$

may be seen as the operation of taking value of the function $a(x_1, \dots, x_n)$ at the point $x_{i_1} = \cdots = x_{i_k} = 0$ where here and later the hat over a symbol means that it is missed.

For each $\alpha \in \mathcal{B}_n$, let $|\alpha| := \alpha_1 + \cdots + \alpha_n$. The ring $\Lambda_n = \bigoplus_{i=0}^n \Lambda_{n,i}$ is a \mathbb{Z} -graded ring ($\Lambda_{n,i} \Lambda_{n,j} \subseteq \Lambda_{n,i+j}$ for all i, j) where $\Lambda_{n,i} := \bigoplus_{|\alpha|=i} Kx^\alpha$. The ideal $\mathfrak{m} := \bigoplus_{i \geq 1} \Lambda_{n,i}$ of Λ_n is called the *augmentation* ideal. Clearly, $K \simeq \Lambda_n / \mathfrak{m}$, $\mathfrak{m}^n = Kx_1 \cdots x_n$ and $\mathfrak{m}^{n+1} = 0$. We say that an element α of \mathcal{B}_n is *even* (resp. *odd*) if the set α contains even (resp. odd) number of elements. By definition, the empty set is even. Let $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$. The ring $\Lambda_n = \Lambda_{n,\bar{0}} \oplus \Lambda_{n,\bar{1}}$ is a \mathbb{Z}_2 -graded ring where $\Lambda_{n,\bar{0}} := \Lambda_n^{ev} := \bigoplus_{\alpha \text{ is even}} Kx^\alpha$ is the subring of even elements of Λ_n and $\Lambda_{n,\bar{1}} := \Lambda_n^{od} := \bigoplus_{\alpha \text{ is odd}} Kx^\alpha$ is the Λ_n^{ev} -module of odd elements of Λ_n . The ring Λ_n has the \mathfrak{m} -adic filtration $\{\mathfrak{m}^i\}_{i \geq 0}$. The even subring Λ_n^{ev} has the induced

\mathfrak{m} -adic filtration $\{\Lambda_{n,\geq i}^{ev} := \Lambda_n^{ev} \cap \mathfrak{m}^i\}$. The Λ_n^{ev} -module Λ_n^{od} has the induced \mathfrak{m} -adic filtration $\{\Lambda_{n,\geq i}^{od} := \Lambda_n^{od} \cap \mathfrak{m}^i\}$.

The K -linear map $a \mapsto \bar{a}$ from Λ_n to itself which is given by the rule

$$\bar{a} := \begin{cases} a, & \text{if } a \in \Lambda_{n,\bar{0}}, \\ -a, & \text{if } a \in \Lambda_{n,\bar{1}}, \end{cases}$$

is a ring automorphism such that $\bar{\bar{a}} = a$ for all $a \in \Lambda_n$. For all $a \in \Lambda_n$ and $i = 1, \dots, n$,

$$x_i a = \bar{a} x_i \quad \text{and} \quad a x_i = x_i \bar{a}. \quad (1)$$

So, each element x_i of Λ_n is a *normal* element, i.e. the two-sided ideal (x_i) generated by the element x_i coincides with both left and right ideals generated by x_i : $(x_i) = \Lambda_n x_i = x_i \Lambda_n$.

For an arbitrary \mathbb{Z} -graded ring $A = \bigoplus_{i \in \mathbb{Z}} A_i$, an additive map $\delta : A \rightarrow A$ is called a *left skew derivation* if

$$\delta(a_i a_j) = \delta(a_i) a_j + (-1)^i a_i \delta(a_j) \quad \text{for all } a_i \in A_i, a_j \in A_j. \quad (2)$$

In this paper, a skew derivation means a *left* skew derivation. Clearly, $1 \in \ker(\delta)$ ($\delta(1) = \delta(1 \cdot 1) = 2\delta(1)$ and so $\delta(1) = 0$). The restriction of the left skew derivation δ to the even subring $A^{ev} := \bigoplus_{i \in 2\mathbb{Z}} A_i$ of A is an ordinary derivation. Recall that an additive subgroup B of A is called a homogeneous subgroup if $B = \bigoplus_{i \in \mathbb{Z}} B \cap A_i$.

Definition. For the ring $\Lambda_n(K)$, consider the set of *left* skew K -derivations:

$$\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$$

given by the rule $\partial_i(x_j) = \delta_{ij}$, the Kronecker delta. Informally, these skew K -derivations will be called (left) *partial skew derivatives*.

Example. $\partial_i(x_1 \cdots x_i \cdots x_k) = (-1)^{i-1} x_1 \cdots x_{i-1} x_{i+1} \cdots x_k$.

If the ring K is commutative, $2 \in K$ is regular (i.e. $2\lambda = 0$ in K implies $\lambda = 0$), and $n \geq 2$, then the centre of $\Lambda_n(K)$ is equal to

$$Z(\Lambda_n) = \begin{cases} \Lambda_{n,\bar{0}}, & \text{if } n \text{ is even,} \\ \Lambda_{n,\bar{0}} \oplus Kx_1 \cdots x_n, & \text{if } n \text{ is odd.} \end{cases}$$

Let K be a commutative ring. For an even n , let $\Lambda_n'^{od} := \Lambda_n^{od}$. For an odd n , let $\Lambda_n'^{od}$ be the K -submodule of Λ_n^{od} generated by all ‘monomials’ x^α but $\theta := x_1 \cdots x_n$, i.e. $\Lambda_n^{od} = \Lambda_n'^{od} \oplus K\theta$. Similarly, for an odd n , let $\Lambda_n'^{ev} := \Lambda_n^{ev}$. For an even n , let $\Lambda_n'^{ev}$ be the K -submodule of Λ_n^{ev} generated by all ‘monomials’ x^α but $\theta := x_1 \cdots x_n$, i.e. $\Lambda_n^{ev} = \Lambda_n'^{ev} \oplus K\theta$. For any n ,

$$\Lambda_n^{od} = \Lambda_n'^{od} \oplus \Lambda_n^{od} \cap Z(\Lambda_n). \quad (3)$$

So, one can naturally identify $\Lambda_n^{od} / \Lambda_n^{od} \cap Z(\Lambda_n)$ with $\Lambda_n'^{od}$.

Consider the sets of *even* and *odd* K -derivations of $\Lambda_n(K)$:

$$\begin{aligned}\mathrm{Der}_K(\Lambda_n)^{ev} &:= \{\delta \in \mathrm{Der}_K(\Lambda_n) \mid \delta(\Lambda_{n,\bar{i}}) \subseteq \Lambda_{n,\bar{i}}, \bar{i} \in \mathbb{Z}_2\}, \\ \mathrm{Der}_K(\Lambda_n)^{od} &:= \{\delta \in \mathrm{Der}_K(\Lambda_n) \mid \delta(\Lambda_{n,\bar{i}}) \subseteq \Lambda_{n,\bar{i}+\bar{1}}, \bar{i} \in \mathbb{Z}_2\}.\end{aligned}$$

So, even derivations are precisely the derivations that respect \mathbb{Z}_2 -grading of Λ_n , and the odd derivations are precisely the derivations that reverse it. The set of odd and even derivations are left $\Lambda_{n,\bar{0}}$ -modules. For each element $a \in \Lambda_n$, one can attach the K -derivation of Λ_n $\mathrm{ad}(a) : b \mapsto [a, b] := ab - ba$, so-called, the *inner* derivation determined by a . The set of all inner derivations is denoted by $\mathrm{IDer}_K(\Lambda_n)$, and the map

$$\Lambda_n / Z(\Lambda_n) \rightarrow \mathrm{IDer}_K(\Lambda_n), \quad a + Z(\Lambda_n) \mapsto \mathrm{ad}(a),$$

is an isomorphism of left $Z(\Lambda_n)$ -modules. The next theorem describes explicitly the sets of all/inner/even and odd derivations.

Theorem 2.1 *Suppose that K is a commutative ring with $\frac{1}{2} \in K$. Then*

1. $\mathrm{Der}_K(\Lambda_n) = \mathrm{Der}_K(\Lambda_n)^{ev} \oplus \mathrm{Der}_K(\Lambda_n)^{od}$.
2. $\mathrm{Der}_K(\Lambda_n)^{ev} = \bigoplus_{i=1}^n \Lambda_n^{od} \partial_i$.
3. $\mathrm{Der}_K(\Lambda_n)^{od} = \mathrm{IDer}_K(\Lambda_n)$ and the map

$$\mathrm{ad} : \Lambda_n^{od} = \Lambda_n^{od} / \Lambda_n^{od} \cap Z(\Lambda_n) \rightarrow \mathrm{IDer}_K(\Lambda_n), \quad a \mapsto \mathrm{ad}(a),$$

is the $Z(\Lambda_n)$ -module isomorphism.

4. $\mathrm{Der}_K(\Lambda_n) / \mathrm{IDer}_K(\Lambda_n) \simeq \mathrm{Der}_K(\Lambda_n)^{ev}$.

Proof. Since $\mathrm{Der}_K(\Lambda_n)^{ev} \cap \mathrm{Der}_K(\Lambda_n)^{od} = 0$, one has the inclusion

$$\mathrm{Der}_K(\Lambda_n) \supseteq \mathrm{Der}_K(\Lambda_n)^{ev} \oplus \mathrm{Der}_K(\Lambda_n)^{od}. \quad (4)$$

Clearly,

$$\mathrm{Der}_K(\Lambda_n)^{ev} \supseteq \sum_{i=1}^n \Lambda_n^{od} \partial_i = \bigoplus_{i=1}^n \Lambda_n^{od} \partial_i, \quad (5)$$

$\mathrm{IDer}_K(\Lambda_n) \simeq \Lambda_n / Z(\Lambda_n) = (\Lambda_n^{ev} \oplus \Lambda_n^{od}) / Z(\Lambda_n) \simeq \Lambda_n^{od} / \Lambda_n^{od} \cap Z(\Lambda_n) \simeq \Lambda_n^{od}$ since $\Lambda_n^{ev} \subseteq Z(\Lambda_n)$. For each $a \in \Lambda_n^{od}$, $\mathrm{ad}(a) \in \mathrm{Der}_K(\Lambda_n)^{od}$, hence

$$\mathrm{Der}_K(\Lambda_n)^{od} \supseteq \mathrm{IDer}_K(\Lambda_n). \quad (6)$$

Note that statement 4 follows from statements 1 and 3. Now, it is obvious that in order to finish the proof of the theorem it suffices to show that

Claim. $\mathrm{Der}_K(\Lambda_n) \subseteq \sum_{i=1}^n \Lambda_n^{od} \partial_i + \mathrm{IDer}_K(\Lambda_n)$.

Indeed, suppose that the inclusion of the claim holds then, by (5) and (6),

$$\text{Der}_K(\Lambda_n) \subseteq \sum_{i=1}^n \Lambda_n^{\text{od}} \partial_i + \text{IDer}_K(\Lambda_n) \subseteq \text{Der}_K(\Lambda_n)^{\text{ev}} \oplus \text{Der}_K(\Lambda_n)^{\text{od}},$$

hence statement 1 is true by (4). Statement 1 together with inclusions (5) and (6) implies statements 2 and 3.

Proof of the Claim. Let δ be a K -derivation of Λ_n . We have to represent the derivation δ as a sum

$$\delta = \sum_{i=1}^n a_i \partial_i + \text{ad}(a), \quad a_i \in \Lambda_n^{\text{od}}, \quad a \in \Lambda_n.$$

The proof of the claim is constructive. According to the decomposition $\Lambda_n = \Lambda_n^{\text{ev}} \oplus \Lambda_n^{\text{od}}$ each element u of Λ_n is a unique sum

$$u = u^{\text{ev}} + u^{\text{od}} \tag{7}$$

of its even and odd components ($u^{\text{ev}} \in \Lambda_n^{\text{ev}}$ and $u^{\text{od}} \in \Lambda_n^{\text{od}}$). For each i , let $u_i := \delta(x_i) = u_i^{\text{ev}} + u_i^{\text{od}}$,

$$\partial := \sum_{i=1}^n u_i^{\text{od}} \partial_i \quad \text{and} \quad \delta' := \delta - \partial.$$

Note that $\partial \in \sum_{i=1}^n \Lambda_n^{\text{od}} \partial_i$, hence changing δ for δ' , if necessary, one may assume that all the elements u_i are even. So, it suffices to show that $\delta = \text{ad}(a)$ for some a . We produce such an a in several steps.

Step 1. Let us prove that, for each $i = 1, \dots, n$, $u_i = v_i x_i$ for some element $v_i \in K[x_1, \dots, \widehat{x}_i, \dots, x_n]^{\text{od}}$. Note that $0 = \delta(0) = \delta(x_i^2) = u_i x_i + x_i u_i = 2u_i x_i$, and so $u_i x_i = 0$ (since $\frac{1}{2} \in K$). This means that $u_i = v_i x_i$ for some element $v_i \in K[x_1, \dots, \widehat{x}_i, \dots, x_n]^{\text{od}}$ since u_i is even. For $n = 1$, it gives $u_1 = 0$ since $K^{\text{od}} = 0$, and we are done. So, let $n \geq 2$.

Step 2. We claim that, for each pair $i \neq j$,

$$v_i|_{x_j=0} = v_j|_{x_i=0}. \tag{8}$$

Evaluating the derivation δ at the element $0 = x_i x_j + x_j x_i$ and taking into account that all the elements u_i are even (hence central) we obtain

$$0 = 2(u_i x_j + u_j x_i) = 2(v_i x_i x_j + v_j x_j x_i) = 2(v_i - v_j) x_i x_j.$$

This means that $v_i - v_j \in (x_i, x_j)$ since $\frac{1}{2} \in K$, or, equivalently, $v_i|_{x_i=0, x_j=0} = v_j|_{x_i=0, x_j=0}$. By Step 1, this equality can be written as (8).

Step 3. Note that $\delta(x_1) = v_1 x_1 = \text{ad}(\frac{1}{2} v_1)(x_1)$ since v_1 is odd, i.e. $(\delta - \text{ad}(\frac{1}{2} v_1))(x_1) = 0$. So, changing δ for $\delta - \text{ad}(\frac{1}{2} v_1)$ one can assume that $\delta(x_1) = 0$, i.e. $v_1 = 0$. Then, by (8), $v_i|_{x_1=0} = 0$ for all $i = 2, \dots, n$, and so $v_i \in (x_1 x_i)$ for all $i = 2, \dots, n$. Summarizing, we can say that by adding to δ a well chosen inner derivation one can assume that $\delta(x_1) = 0$ and $\delta(x_i) \in (x_1 x_i)$ for all $i \geq 2$. This statement serves as the base of the induction in the

proof of the next statement. For each k such that $1 \leq k \leq n$, by adding to δ a certain inner derivation we can assume that

$$\delta(x_1) = \cdots = \delta(x_k) = 0, \quad \delta(x_i) \in (x_1 \cdots x_k x_i), \quad k < i \leq n. \quad (9)$$

So, assuming that (9) holds for k we must prove the same statement but for $k+1$. Note that $v_{k+1}x_{k+1} = \delta(x_{k+1}) \in (x_1 \cdots x_k)$, hence $v_{k+1} = x_1 \cdots x_k v$ for some $v \in K[x_{k+2}, \dots, x_n]$. Consider the derivation $\delta' := \delta - \text{ad}(\frac{1}{2}v_{k+1})$. For each $i = 1, \dots, k$, $\delta'(x_i) = \delta(x_i) = 0$ as $v_{k+1} \in (x_1 \cdots x_k)$; and $\delta'(x_{k+1}) = v_{k+1}x_{k+1} - [\frac{1}{2}v_{k+1}, x_{k+1}] = v_{k+1}x_{k+1} - v_{k+1}x_{k+1} = 0$. These prove the first part of (9) for $k+1$, namely, that

$$\delta'(x_1) = \cdots = \delta'(x_{k+1}) = 0.$$

So, changing δ for δ' one can assume that

$$\delta(x_1) = \cdots = \delta(x_{k+1}) = 0.$$

These conditions imply that $v_1 = \cdots = v_{k+1} = 0$. If $n = k+1$, we are done. So, let $k+1 < n$. Then, by (8), for each $i > k+1$, $v_i \in \cap_{j=1}^{k+1} (x_j) = (x_1 \cdots x_{k+1})$, hence $\delta(x_i) = v_i x_i \in (x_1 \cdots x_{k+1} x_i)$. By induction, (9) is true for all k . In particular, for $k = n$ one has $\delta = 0$. This means that δ is an inner derivation, as required. \square

The ring $K[x]/(x^2)$ of dual numbers is the Grassmann ring Λ_1 .

Corollary 2.2 *Suppose that K is a commutative ring with $\frac{1}{2} \in K$. Then $\text{Der}_K(K[x]/(x^2)) = \text{Der}_K(K[x]/(x^2))^{ev} = Kx \frac{d}{dx}$ and $\text{Der}_K(K[x]/(x^2))^{od} = 0$ where $\frac{d}{dx}$ is the skew K -derivation of $K[x]/(x^2)$.*

A Lie algebra $(\mathcal{G}, [\cdot, \cdot])$ over K is positively graded if $\mathcal{G} = \oplus_{i \geq 0} \mathcal{G}_i$ is a direct sum of K -submodules such that $[\mathcal{G}_i, \mathcal{G}_j] \subseteq \mathcal{G}_{i+j}$ for all $i, j \geq 0$.

$(\text{Der}_K(\Lambda_n), [\cdot, \cdot])$ is a Lie algebra over K where $[\delta, \partial] := \delta\partial - \partial\delta$. By Theorem 2.1, the Lie algebra $\text{Der}_K(\Lambda_n) = \oplus_{i \geq 0} D_i$ is a positively graded Lie algebra where

$$D_i := \{\delta \in \text{Der}_K(\Lambda_n) \mid \delta(\Lambda_{n,j}) \subseteq \Lambda_{n,j+i}, \quad j \geq 0\}.$$

Clearly, $D_i = 0$, $i \geq n$. For each even natural number i such that $0 \leq i \leq n-1$,

$$D_i = \oplus_{j=1}^n \Lambda_{n,i+1} \partial_j. \quad (10)$$

For each odd natural number i such that $1 \leq i \leq n-1$,

$$D_i = \{\text{ad}(a) \mid a \in \Lambda_{n,i}\} \simeq \Lambda_{n,i}, \quad \text{ad}(a) \mapsto a. \quad (11)$$

The zero component $D_0 = \oplus_{i,j=0}^n Kx_i \partial_j$ of $\text{Der}_K(\Lambda_n)$ is a Lie subalgebra of $\text{Der}_K(\Lambda_n)$ which is canonically isomorphic to the Lie algebra $\text{gl}_n(K) := \oplus_{i,j=1}^n K E_{ij}$ via $D_0 \rightarrow \text{gl}_n(K)$, $x_i \partial_j \mapsto E_{ij}$, where E_{ij} are the matrix units. By the very definition, $D_+ := \oplus_{i \geq 1} D_i$ is a nilpotent ideal of the Lie algebra $\text{Der}_K(\Lambda_n)$ such that $\text{Der}_K(\Lambda_n) = D_0 \oplus D_+ \simeq \text{gl}_n(K) \oplus D_+$

and $\text{Der}_K(\Lambda_n)/D_+ \simeq \mathfrak{gl}_n(K)$. So, if K is a field of characteristic zero then D_+ is the radical of the Lie algebra $\text{Der}_K(\Lambda_n)$.

The Lie algebra $\text{Der}_K(\Lambda_n) = \text{Der}_K(\Lambda_n)^{ev} \oplus \text{Der}_K(\Lambda_n)^{od}$ is a \mathbb{Z}_2 -graded Lie algebra.

By Theorem 2.1, any K -derivation δ of Λ_n is a unique sum $\delta = \delta^{ev} + \delta^{od}$ of even and odd derivations, and $\delta^{od} := -\frac{1}{2}\text{ad}(a)$ for a *unique* element $a \in \Lambda_n^{od}$. In order to find the element a (Corollary 2.5), we need two theorems which are interesting on their own right. Theorem 2.3 gives a unique (sort of ‘triangular’) canonical presentation of any element of Λ_n . This presentation is important in dealing with derivations and skew derivations. The element a in $\delta^{od} = -\frac{1}{2}\text{ad}(a)$ is given in this form (Corollary 2.5). In order to find the element a we need to find solutions to the system of equations (Theorem 2.4). This system is a kind of Poincaré Lemma for the (noncommutative) Grassmann algebra Λ_n .

Theorem 2.3 [1] *Let K be an arbitrary (not necessarily commutative) ring. Then*

1. *the Grassmann ring $\Lambda_n(K)$ is a direct sum of right K -modules*

$$\begin{aligned} \Lambda_n(K) = & x_1 \cdots x_n K \oplus x_1 \cdots x_{n-1} K \oplus x_1 \cdots x_{n-2} K[x_n] \oplus \cdots \\ & \cdots \oplus x_1 \cdots x_i K[x_{i+2} \dots, x_n] \oplus \cdots \oplus x_1 K[x_3 \dots, x_n] \oplus K[x_2 \dots, x_n]. \end{aligned}$$

2. *So, each element $a \in \Lambda_n(K)$ is a unique sum*

$$a = x_1 \cdots x_n a_n + x_1 \cdots x_{n-1} b_n + \sum_{i=1}^{n-2} x_1 \cdots x_i b_{i+1} + b_1$$

where $a_n, b_n \in K$, $b_i \in K[x_{i+1} \dots, x_n]$, $1 \leq i \leq n-1$. Moreover,

$$\begin{aligned} a_n &= \partial_n \partial_{n-1} \cdots \partial_1(a), \\ b_{i+1} &= \partial_i \partial_{i-1} \cdots \partial_1(1 - x_{i+1} \partial_{i+1})(a), \quad 1 \leq i \leq n-1, \\ b_1 &= (1 - x_1 \partial_1)(a). \end{aligned}$$

So,

$$a = x_1 \cdots x_n \partial_n \partial_{n-1} \cdots \partial_1(a) + \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1(1 - x_{i+1} \partial_{i+1})(a) + (1 - x_1 \partial_1)(a).$$

By Theorem 2.3, the identity map $\text{id}_{\Lambda_n} : \Lambda_n \rightarrow \Lambda_n$ is equal to

$$\text{id}_{\Lambda_n} = x_1 \cdots x_n \partial_n \partial_{n-1} \cdots \partial_1 + \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1(1 - x_{i+1} \partial_{i+1}) + (1 - x_1 \partial_1). \quad (12)$$

Theorem 2.4 [1] *Let K be an arbitrary ring, $u_1, \dots, u_n \in \Lambda_n(K)$, and $a \in \Lambda_n(K)$ be an unknown. Then the system of equations*

$$\begin{cases} x_1 a = u_1 \\ x_2 a = u_2 \\ \vdots \\ x_n a = u_n \end{cases}$$

has a solution in Λ_n iff the following two conditions hold

1. $u_1 \in (x_1), \dots, u_n \in (x_n)$, and
2. $x_i u_j = -x_j u_i$ for all $i \neq j$.

In this case,

$$a = x_1 \cdots x_n a_n + \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(u_{i+1}) + \partial_1(u_1), \quad a_n \in K, \quad (13)$$

are all the solutions.

The next corollary describes explicitly δ^{ev} and δ^{od} in $\delta = \delta^{ev} + \delta^{od}$.

Corollary 2.5 *Let K be a commutative ring with $\frac{1}{2} \in K$, δ be a K -derivation of $\Lambda_n(K)$, and, for each $i = 1, \dots, n$, $\delta(x_i) = u_i^{ev} + u_i^{od}$ for unique elements $u_i^{ev} \in \Lambda_n^{ev}$ and $u_i^{od} \in \Lambda_n^{od}$. Then*

1. $\delta^{ev} = \sum_{i=1}^n u_i^{od} \partial_i$, and
2. $\delta^{od} = -\frac{1}{2} \text{ad}(a)$ where the unique element $a \in \Lambda_n^{od}$ is given by the formula

$$a = \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(u_{i+1}^{ev}) + \partial_1(u_1^{ev}).$$

Proof. 1. This statement has been proved already in the proof of Theorem 2.1.

2. For each $i = 1, \dots, n$, on the one hand $\delta^{od}(x_i) = (\delta - \delta^{ev})(x_i) = u_i^{ev}$; on the other, $\delta^{ev}(x_i) = -\frac{1}{2}(ax_i - x_i a) = \frac{1}{2}2x_i a = x_i a$. So, the element a is a solution to the system of equations

$$\begin{cases} x_1 a = u_1^{ev} \\ x_2 a = u_2^{ev} \\ \vdots \\ x_n a = u_n^{ev}. \end{cases}$$

By Theorem 2.4 and the fact that $a \in \Lambda_n'^{od}$ (i.e. $a_n = 0$), we have

$$a = \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(u_{i+1}^{ev}) + \partial_1(u_1^{ev}). \quad \square$$

Let K be a field. Let V be a finite dimensional vector space over K and $a \in \text{End}_K(V)$, a K -linear map on V . The vector space V is the $K[t]$ -module where $t \cdot v := av$; V is the $K[a]$ -module for short. The linear map a is called *semi-simple* (resp. *nilpotent*) if the $K[a]$ -module V is semi-simple (resp. $a^k = 0$ for some $k \geq 1$). It is well-known that a is a unique sum $a = a_s + a_n$ where a_s is a semi-simple map, a_n is a nilpotent map, and $a_s, a_n \in K[a] := \sum_{i \geq 0} K a^i$ (in particular, the maps a , a_s , and a_n commute). If V is a finite dimensional K -algebra and a is a K -derivation of the algebra V then the maps a_s and a_n are also K -derivations.

The subsets of $\text{Der}_K(V)$ of all semi-simple derivations $\text{Der}_K(V)_s$ and all nilpotent derivations $\text{Der}_K(V)_n$ do not meet, i.e. $\text{Der}_K(V)_s \cap \text{Der}_K(V)_n = 0$. In general, the sets of semi-simple and nilpotent derivations are *not* vector spaces, though they are closed under scalar multiplication.

Let K be a commutative ring with $\frac{1}{2} \in K$. The following K -derivations of $\Lambda_n = \Lambda_n(K)$

$$h_1 := x_1 \partial_1, \dots, h_n := x_n \partial_n,$$

commute, $h_1^2 = h_1, \dots, h_n^2 = h_n$, and $h_i(x^\alpha) = \alpha_i x^\alpha$ for all i and α . Since $h_i(x_j) = \delta_{ij} x_j$, the maps h_1, \dots, h_n are linearly independent. So, h_1, \dots, h_n are *commuting, semi-simple, K -linearly independent, idempotent K -derivations* of the algebra Λ_n . For each $i = 1, \dots, n$, $\Lambda_n = K_i \oplus x_i K_i$ where $K_i := \ker(\partial_i) = K[x_1, \dots, \widehat{x_i}, \dots, x_n]$, and $h_i : \Lambda_n \rightarrow \Lambda_n$ is the projection onto $x_i K_i$.

Let H be the subalgebra of the endomorphism algebra $\text{End}_K(\Lambda_n)$ generated by the elements h_1, \dots, h_n . As an abstract algebra $H \simeq K[H_1, \dots, H_n]/(H_1^2, \dots, H_n^2)$. The algebra $\Lambda_n = \bigoplus_{\alpha \in \mathcal{B}_n} K x^\alpha$ is a semi-simple H -module where each isotypic component is simple: $K x^\alpha$ is the simple H -module, and $K x^\alpha \simeq K x^\beta$ as H -modules iff $\alpha = \beta$. Let

$$\begin{aligned} \text{Der}_K(\Lambda_n)_s^{ev} &:= \text{Der}_K(\Lambda_n)^{ev} \cap \text{Der}(\Lambda_n)_s & \text{Der}_K(\Lambda_n)_n^{ev} &:= \text{Der}_K(\Lambda_n)^{ev} \cap \text{Der}(\Lambda_n)_n, \\ \text{Der}_K(\Lambda_n)_s^{od} &:= \text{Der}_K(\Lambda_n)^{od} \cap \text{Der}(\Lambda_n)_s & \text{Der}_K(\Lambda_n)_n^{od} &:= \text{Der}_K(\Lambda_n)^{od} \cap \text{Der}(\Lambda_n)_n. \end{aligned}$$

Definition. An ideal \mathfrak{a} of Λ_n is called a *differential ideal* (or a $\text{Der}_K(\Lambda_n)$ -invariant ideal) if $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ for all $\delta \in \text{Der}_K(\Lambda_n)$.

The next proposition describes all the differential ideals of $\Lambda_n(K)$.

Proposition 2.6 *Let K be a commutative ring with $\frac{1}{2} \in K$, $\mathcal{F}_n(K) := \{I : I_0 \subseteq I_1 \subseteq \dots \subseteq I_n \mid I_i \text{ are ideals of } K\}$ be the set of n -flags of ideals of K , $\text{DI}(\Lambda_n)$ be the set of all differentiable ideals of $\Lambda_n(K)$. Then the map*

$$\mathcal{F}_n(K) \rightarrow \text{DI}(\Lambda_n), \quad I \mapsto \widehat{I} := \bigoplus_{i=0}^n \bigoplus_{|\alpha|=i} I_i x^\alpha,$$

is a bijection. In particular, \mathfrak{m}^i , $0 \leq i \leq n+1$, are differential ideals of Λ_n ; these are the only differential ideals of Λ_n if K is a field of characteristic $\neq 2$.

Proof. Recall that $\text{Der}_K(\Lambda_n) = \bigoplus_{i \geq 0} D_i$. By (10) and (11), the map $I \mapsto \widehat{I}$ is well-defined and injective, by the very definition. It remains to show that each differential idea, say \mathfrak{a} , of $\Lambda_n(K)$ is equal to \widehat{I} for some $I \in \mathcal{F}_n(K)$. Since $\sum_{i=1}^n Kh_i \subseteq \text{Der}_K(\Lambda_n)$ and $\Lambda_n = \bigoplus_{\alpha \in \mathcal{B}_n} Kx^\alpha$ is the direct sum of non-isomorphic simple H -modules, we have

$$\mathfrak{a} = \bigoplus_{\alpha \in \mathcal{B}_n} (\mathfrak{a} \cap Kx^\alpha) = \bigoplus_{\alpha \in \mathcal{B}_n} \mathfrak{a}_\alpha x^\alpha$$

where \mathfrak{a}_α is an ideal of K such that $\mathfrak{a}_\alpha x^\alpha = \mathfrak{a} \cap Kx^\alpha$. For each i such that $0 \leq i \leq n$, $\bigoplus_{|\alpha|=i} \mathfrak{a}_\alpha x^\alpha$ is a D_0 -module where $D_0 = \bigoplus_{i,j=1}^n Kx_i \partial_j$, hence all the ideals \mathfrak{a}_α coincide where $|\alpha| = i$. Let I_i be their common value. Since \mathfrak{a} is an ideal of Λ_n , $\{I : I_0 \subseteq \cdots \subseteq I_n\} \in \mathcal{F}_n(K)$, and so $\mathfrak{a} = \widehat{I}$, as required. \square

Definition. A ring R is called a *differentiably simple* ring if it is a simple left $\text{Der}(R)$ -module.

So, the algebra Λ_n is *not* differentiably simple if $n \geq 1$ where K is a commutative ring with $\frac{1}{2} \in K$.

Let K be a *reduced* commutative ring with $\frac{1}{2} \in K$. Let \mathcal{S} be the set of all n -tuples (s_1, \dots, s_n) where s_1, \dots, s_n are commuting, idempotent K -derivations (i.e. $s_i^2 = s_i$) of Λ_n such that the following conditions hold: $\bigcap_{i=1}^n \ker(s_i) = K$; all the K -modules $\mathcal{K}_i := \mathfrak{m} \cap \ker(s_i - 1) \cap \bigcap_{j \neq i} \ker(s_j)$ are free of rank 1 over K , i.e. $\mathcal{K}_i = Kx'_i \simeq K$ for some element $x'_i \in \mathfrak{m}$; $\Lambda_n = \ker(s_1) + \mathcal{K}_1 \ker(s_1)$; and, for each $i = 1, \dots, n-1$,

$$K_{1,\dots,i} = K_{1,\dots,i+1} + \mathcal{K}_{i+1} K_{1,\dots,i+1} \quad (14)$$

where $K_{1,\dots,i} := \bigcap_{j=1}^i \ker(s_j)$.

Clearly, $(h_1, \dots, h_n) \in \mathcal{S}$ (see (17) below). For each $(s_1, \dots, s_n) \in \mathcal{S}$, the maps s_1, \dots, s_n are K -linearly independent ($\sum \mu_i s_i = 0 \Rightarrow 0 = (\sum \mu_i s_i)(\mathcal{K}_i) = \mu_i Kx'_i \Rightarrow \mu_i = 0$).

Let $G := \text{Aut}_K(\Lambda_n)$ be the group of K -algebra automorphisms of the Grassmann algebra Λ_n . For $\sigma \in G$, let $x'_i := \sigma(x_i)$. Then $x_i'^2 = \sigma(x_i^2) = \sigma(0) = 0$. If $\lambda_i \equiv x'_i \pmod{\mathfrak{m}}$ for some $\lambda_i \in K$ then $\lambda_i^2 = 0$, hence $\lambda_i = 0$ since K is reduced. Therefore, $\sigma(\mathfrak{m}) = \mathfrak{m}$, and so

$$\sigma(\mathfrak{m}^i) = \mathfrak{m}^i \text{ for all } i \geq 1. \quad (15)$$

By (15), the group G acts on the set \mathcal{S} by conjugation (i.e. by changing generators): $\sigma \cdot (s_1, \dots, s_n) := (\sigma s_1 \sigma^{-1}, \dots, \sigma s_n \sigma^{-1})$. We prove shortly that the group G act transitively on the set \mathcal{S} (Corollary 2.9.(2)) and the stabilizer $\text{St}(h_1, \dots, h_n)$ of the element (h_1, \dots, h_n) is equal to the '*n-dimensional algebraic torus*' (where K^* is the group of units of K)

$$\mathbb{T}^n := \{\sigma_\lambda \mid \lambda \in K^{*n}, \sigma_\lambda(x_i) = \lambda_i x_i, 1 \leq i \leq n\} \simeq K^{*n} \quad (\text{Lemma 2.7}).$$

Therefore,

$$\mathcal{S} = G \cdot (h_1, \dots, h_n) \simeq G / \mathbb{T}^n. \quad (16)$$

Note that $\text{St}(h_1, \dots, h_n) = \{\sigma \in G \mid \sigma h_i = h_i \sigma, 1 \leq i \leq n\}$.

Lemma 2.7 *Let K be a reduced commutative ring with $\frac{1}{2} \in K$. Then $\text{St}(h_1, \dots, h_n) = \mathbb{T}^n$.*

Proof. Clearly, the torus is a subgroup of the stabilizer. We have to show that each element σ of the stabilizer belongs to the torus. Since the automorphism σ commutes with all the h_i , the automorphism σ respects eigenspaces of h_i (i.e. $\ker(h_i)$ and $\ker(h_i - 1)$) and their intersections. In particular, for each $i = 1, \dots, n$, the vector space

$$\ker(h_1) \cap \dots \cap \ker(h_{i-1}) \cap \ker(h_i - 1) \cap \ker(h_{i+1}) \cap \dots \cap \ker(h_n) = Kx_i \quad (17)$$

is σ -invariant, i.e. $\sigma(x_i) = \lambda_i x_i$ for some $\lambda_i \in K^*$, and so $\sigma \in \mathbb{T}^n$. \square

Let \mathcal{A} be the set of all the n -tuples (x'_1, \dots, x'_n) of canonical generators for the K -algebra $\Lambda_n(K)$ ($x_i'^2 = 0$ and $x'_i x'_j = -x'_j x'_i$). Clearly, all $x'_i \in \mathfrak{m}$ since K is reduced. The group G acts on the set \mathcal{A} in the obvious way: $\sigma(x'_1, \dots, x'_n) = (\sigma(x'_1), \dots, \sigma(x'_n))$. The action is transitive and the stabilizer of each point is trivial (by the very definition of the group G).

Lemma 2.8 *Let K be a reduced commutative ring with $\frac{1}{2} \in K$, $(s_1, \dots, s_n) \in \mathcal{S}$, and $\mathcal{K}_i = Kx'_i$, $1 \leq i \leq n$. Then the element (x'_1, \dots, x'_n) belongs to the set \mathcal{A} , and $s_1 = x'_1 \frac{\partial}{\partial x'_1}, \dots, s_n = x'_n \frac{\partial}{\partial x'_n}$.*

Proof. By Proposition 2.6, $\delta(\mathfrak{m}) \subseteq \mathfrak{m}$ for any K -derivation δ of Λ_n . In particular, $s_1(\mathfrak{m}) \subseteq \mathfrak{m}, \dots, s_n(\mathfrak{m}) \subseteq \mathfrak{m}$. Using $n - 1$ times (14), we have

$$\begin{aligned} \Lambda_n &= K_1 + x'_1 K_1 = (K_{1,2} + x'_2 K_{1,2}) + x'_1 (K_{1,2} + x'_2 K_{1,2}) \\ &= K_{1,2} + x'_2 K_{1,2} + x'_1 K_{1,2} + x'_1 x'_2 K_{1,2} = \dots \\ &= \sum_{\alpha \in \mathcal{B}_n} x'^{\alpha} K_{1,\dots,n} = \sum_{\alpha \in \mathcal{B}_n} x'^{\alpha} K = \sum_{\alpha \in \mathcal{B}_n} K x'^{\alpha} \end{aligned}$$

since $K_{1,\dots,n} = K$. Since $\Lambda_n(K)$ is a free module of rank 2^n over the commutative ring K with identity, $x'^{\alpha} \neq 0$ for all α , and the sums above are the direct sums, i.e. $\Lambda_n = \bigoplus_{\alpha \in \mathcal{B}_n} K x'^{\alpha}$.

For each i and α , $s_i(x'^{\alpha}) = \alpha_i x'^{\alpha}$ and $s_i(y_{\alpha,\nu}) = \alpha_i y_{\alpha,\nu}$ where $y_{\alpha,\nu} := x'^{\alpha_{\nu(1)}}_{\nu(1)} \dots x'^{\alpha_{\nu(n)}}_{\nu(n)}$ and $\nu \in S_n$ (S_n is the symmetric group). Therefore, $K x'^{\alpha} := K x'^{\alpha_1}_1 \dots x'^{\alpha_n}_n = K x'^{\alpha_{\nu(1)}}_{\nu(1)} \dots x'^{\alpha_{\nu(n)}}_{\nu(n)}$ for any permutation $\nu \in S_n$ (since the sums above are direct). In particular, for each $i \neq j$, $x'_i x'_j = \lambda x'_j x'_i$ for some $\lambda = \lambda_{ij} \in K$. We claim that $\lambda = -1$. For, note that $\mathfrak{m} = (x'_1, \dots, x'_n)$ and so the set $\bar{x}'_1 := x'_1 + \mathfrak{m}^2, \dots, \bar{x}'_n := x'_n + \mathfrak{m}^2$ is a basis for the vector space $\mathfrak{m}/\mathfrak{m}^2$ over K . In $\mathfrak{m}^2/\mathfrak{m}^3$, on the one hand, $\bar{x}'_i \bar{x}'_j = -\bar{x}'_j \bar{x}'_i \neq 0$; on the other, by taking the equation $x'_i x'_j = \lambda x'_j x'_i$ modulo \mathfrak{m}^3 , we have $\bar{x}'_i \bar{x}'_j = \lambda \bar{x}'_j \bar{x}'_i$; hence $\lambda = -1$, as required.

For each i , $x_i'^2 \in \ker(s_i)$ (since $s_i(x_i'^2) = 2x_i'^2$ and 2 is not an eigenvalue for the idempotent derivation s_i as $\frac{1}{2} \in K$), hence $x_i'^2 \in \mathfrak{m} \cap \bigcap_{i=1}^n \ker(s_i) = \mathfrak{m} \cap K = 0$, i.e. $x_i'^2 = 0$. This proves that the elements x'_1, \dots, x'_n are *canonical* generators for the algebra Λ_n . Now, it is obvious that $s_1 = x'_1 \frac{\partial}{\partial x'_1}, \dots, s_n = x'_n \frac{\partial}{\partial x'_n}$. \square

If a group \mathcal{G} acts on a set X we say that X is a \mathcal{G} -set. Let Y be a \mathcal{G} -set. A map $f : X \rightarrow Y$ is called a \mathcal{G} -map if $f(gx) = gf(x)$ for all $x \in X$ and $g \in \mathcal{G}$. A \mathcal{G} -isomorphism is a \mathcal{G} -map which is a bijection. The torus \mathbb{T}^n acts on the set \mathcal{A} by the rule $(\lambda_i)(x'_i) := (\lambda_i x'_i)$. Let \mathcal{A}/\mathbb{T}^n be the set of all \mathbb{T}^n -orbits.

Corollary 2.9 *Let K be a reduced commutative ring with $\frac{1}{2} \in K$. Then*

1. *The map $\mathcal{S} \rightarrow \mathcal{A}/\mathbb{T}^n$, $(s_1, \dots, s_n) \mapsto \mathbb{T}^n(x'_1, \dots, x'_n)$, is a G -isomorphism with the inverse $\mathbb{T}^n(x'_1, \dots, x'_n) \mapsto (x'_1 \frac{\partial}{\partial x'_1}, \dots, x'_n \frac{\partial}{\partial x'_n})$.*
2. *In particular, G acts transitively on the set \mathcal{S} .*

Proof. 1. This follows directly from Lemma 2.8.

2. The group G acts transitively on the set \mathcal{A} , hence it does on the set \mathcal{S} , by statement 1. \square

For $n = 1$, Corollary 2.9 gives $\mathcal{S} = \{x_1 \partial_1\}$ since $G = \mathbb{T}$.

Let \mathcal{G} be a group and M be a \mathcal{G} -module. We say that M is a *faithful* \mathcal{G} -module (or the group \mathcal{G} acts *faithfully* on M) if the map $\mathcal{G} \rightarrow \text{End}(M)$ is injective.

Theorem 2.10 *Let K be a reduced commutative ring with $\frac{1}{2} \in K$, $n \geq 1$. Then*

1. *$\text{Der}_K(\Lambda_n)$ is a faithful G -module iff $n \geq 2$.*
2. *The G -module $\text{Der}_K(\Lambda_n)$ is not simple.*

Proof. 1. For $n = 1$, $\text{Der}_K(\Lambda_1) = Kx_1 \partial_1$ (Corollary 2.2) and $G = \mathbb{T} = \text{St}(x_1 \partial_1)$. Therefore, $\text{Der}_K(\Lambda_1)$ is not a faithful G -module. So, let $n \geq 2$. Suppose that an element $\sigma \in G$ acts trivially on $\text{Der}_K(\Lambda_n)$, i.e. $\sigma \delta \sigma^{-1} = \delta$ for all $\delta \in \text{Der}_K(\Lambda_n)$. We have to show that $\sigma = e$, the identity element of G . By Lemma 2.7, $\sigma = \sigma_\lambda \in \mathbb{T}^n$ for some $\lambda \in K^{*n}$. For each $i = 1, \dots, n$, $\text{ad}(x_i) = \sigma \text{ad}(x_i) \sigma^{-1} = \text{ad}(\sigma(x_i)) = \lambda_i \text{ad}(x_i)$, hence $\lambda_i = 1$ (choose j such that $j \neq i$; then $0 = (\lambda_i - 1) \text{ad}(x_i)(x_j) = 2(\lambda_i - 1)x_i x_j$, and so $\lambda_i - 1 = 0$), i.e. $\sigma = e$.

2. If $n \geq 1$, then the G -module $\text{Der}_K(\Lambda_n) = D_0 \oplus D_+$ contains the proper submodule D_+ , and so $\text{Der}_K(\Lambda_n)$ is not a simple G -module. \square

3 Skew derivations of the Grassmann rings

Let K be a commutative ring. Recall that the Grassmann K -algebra $\Lambda_n = \Lambda_{n, \bar{0}} \oplus \Lambda_{n, \bar{1}}$ is a \mathbb{Z}_2 -graded algebra, $\Lambda_{n, \bar{0}} = \Lambda_n^{ev}$ and $\Lambda_{n, \bar{1}} = \Lambda_n^{od}$. Each element a of Λ_n is a unique sum $a = a_{\bar{0}} + a_{\bar{1}}$ with $a_{\bar{0}} \in \Lambda_{n, \bar{0}}$ and $a_{\bar{1}} \in \Lambda_{n, \bar{1}}$. We also use the alternative notation: $a = a^{ev} + a^{od}$ where $a^{ev} := a_{\bar{0}}$ and $a^{od} := a_{\bar{1}}$.

Recall that a K -linear map $\delta : \Lambda_n \rightarrow \Lambda_n$ is called a (*left*) *skew K -derivation*, if for any $b_s \in \Lambda_{n, s}$ and $b_t \in \Lambda_{n, t}$ (where $s, t \in \mathbb{Z}_2$),

$$\delta(b_s b_t) = \delta(b_s) b_t + (-1)^s b_s \delta(b_t).$$

The set of all skew derivations $\text{SDer}_K(\Lambda_n)$ is a left $Z(\Lambda_n)$ -module and a left Λ_n^{ev} -module since $\Lambda_n^{ev} \subseteq Z(\Lambda_n)$.

Consider the sets of even and odd skew K -derivations of $\Lambda_n(K)$:

$$\begin{aligned} \text{SDer}_K(\Lambda_n)^{ev} &:= \{ \delta \in \text{SDer}_K(\Lambda_n) \mid \delta(\Lambda_{n, \bar{i}}) \subseteq \Lambda_{n, \bar{i}}, \bar{i} \in \mathbb{Z}_2 \}, \\ \text{SDer}_K(\Lambda_n)^{od} &:= \{ \delta \in \text{SDer}_K(\Lambda_n) \mid \delta(\Lambda_{n, \bar{i}}) \subseteq \Lambda_{n, \bar{i}+\bar{1}}, \bar{i} \in \mathbb{Z}_2 \}. \end{aligned}$$

So, even skew derivations are precisely the skew derivations that respect \mathbb{Z}_2 -grading of Λ_n , and the odd skew derivations are precisely the skew derivations that reverse it. The set of odd and even skew derivations are left $\Lambda_{n,\bar{0}}$ -modules. For each element $a \in \Lambda_n$, one can attach, so-called, the *inner* skew K -derivation of Λ_n : $\text{sad}(a) : b_s \mapsto ab_s - (-1)^s b_s a$, where $b_s \in \Lambda_{n,s}$, $s \in \mathbb{Z}_2$. The set of all inner skew derivations is denoted by $\text{ISDer}_K(\Lambda_n)$. The kernel of the K -linear map $\text{sad} : \Lambda_n \rightarrow \text{ISDer}_K(\Lambda_n)$, $a \mapsto \text{sad}(a)$, is equal to $\ker(\text{sad}) = \Lambda_n^{od} + Kx_1 \cdots x_n$, and

$$\Lambda_n / \ker(\text{sad}) = \Lambda_n^{ev} \oplus \Lambda_n^{od} / (\Lambda_n^{od} + Kx_1 \cdots x_n) \simeq \Lambda_n^{ev} / \Lambda_n^{ev} \cap Kx_1 \cdots x_n \simeq \Lambda_n^{ev}.$$

By the Homomorphism Theorem, the map

$$\Lambda_n^{ev} \rightarrow \text{ISDer}_K(\Lambda_n), \quad a \mapsto \text{sad}(a), \quad (18)$$

is a bijection and $\text{ISDer}_K(\Lambda_n) = \{\text{sad}(a) \mid a \in \Lambda_n^{ev}\}$.

The next theorem describes explicitly the sets of all/inner/even and odd skew derivations.

Theorem 3.1 *Suppose that K is a commutative ring with $\frac{1}{2} \in K$. Then*

1. $\text{SDer}_K(\Lambda_n) = \text{SDer}_K(\Lambda_n)^{ev} \oplus \text{SDer}_K(\Lambda_n)^{od}$.
2. $\text{SDer}_K(\Lambda_n)^{od} = \bigoplus_{i=1}^n \Lambda_n^{ev} \partial_i$.
3. $\text{SDer}_K(\Lambda_n)^{ev} = \text{ISDer}_K(\Lambda_n) = \{\text{sad}(a) \mid a \in \Lambda_n^{ev}\}$, and the map $\text{sad} : \Lambda_n^{ev} \rightarrow \text{ISDer}_K(\Lambda_n)$, $a \mapsto \text{sad}(a)$, is a bijection.
4. $\text{SDer}_K(\Lambda_n) / \text{ISDer}_K(\Lambda_n) \simeq \text{SDer}_K(\Lambda_n)^{od}$.

Proof. By (18), the map in statement 3 is a bijection and $\text{ISDer}_K(\Lambda_n) = \{\text{sad}(a) \mid a \in \Lambda_n^{ev}\}$. Since $\text{SDer}_K(\Lambda_n)^{ev} \cap \text{SDer}_K(\Lambda_n)^{od} = 0$, one has the inclusion

$$\text{SDer}_K(\Lambda_n) \supseteq \text{SDer}_K(\Lambda_n)^{ev} \oplus \text{SDer}_K(\Lambda_n)^{od}. \quad (19)$$

Clearly,

$$\text{SDer}_K(\Lambda_n)^{od} \supseteq \sum_{i=1}^n \Lambda_n^{ev} \partial_i = \bigoplus_{i=1}^n \Lambda_n^{ev} \partial_i. \quad (20)$$

For each $a \in \Lambda_n^{ev}$, $\text{sad}(a) \in \text{SDer}_K(\Lambda_n)^{ev}$, hence

$$\text{SDer}_K(\Lambda_n)^{ev} \supseteq \text{ISDer}_K(\Lambda_n). \quad (21)$$

Note that statement 4 follows from statements 1 and 3. Now, it is obvious that in order to finish the proof of the theorem it suffices to prove the next claim.

Claim. $\text{SDer}_K(\Lambda_n) \subseteq \sum_{i=1}^n \Lambda_n^{ev} \partial_i + \text{ISDer}_K(\Lambda_n)$.

Indeed, suppose that the inclusion of the claim holds then

$$\text{SDer}_K(\Lambda_n) \subseteq \sum_{i=1}^n \Lambda_n^{ev} \partial_i + \text{ISDer}_K(\Lambda_n) \subseteq \text{SDer}_K(\Lambda_n)^{ev} \oplus \text{SDer}_K(\Lambda_n)^{od},$$

hence statement 1 is true by (19). Statement 1 together with inclusions (20) and (21) implies statements 2 and 3.

Proof of the Claim. Let δ be a skew K -derivation of Λ_n . We have to represent the derivation δ as a sum

$$\delta = \sum_{i=1}^n a_i \partial_i + \text{sad}(a), \quad a_i \in \Lambda_n^{ev}, \quad a \in \Lambda_n^{od}.$$

The proof of the claim is constructive. By (7), for each $i = 1, \dots, n$, let

$$u_i := \delta(x_i) = u_i^{ev} + u_i^{od}, \quad \text{where } u_i^{ev} \in \Lambda_n^{ev}, \quad u_i^{od} \in \Lambda_n^{od}. \quad (22)$$

Let $\partial := \sum_{i=1}^n u_i^{ev} \partial_i$ and $\delta' := \delta - \partial$. Then $\delta'(x_i) = u_i^{od}$. Note that $\partial \in \sum_{i=1}^n \Lambda_n^{ev} \partial_i$. Then changing δ for δ' , if necessary, one may assume that all u_i belong to Λ_n^{od} . Now, it suffices to show that $\delta = \text{sad}(a)$ for some $a \in \Lambda_n^{ev}$. We construct such an a in several steps.

Step 1. Let us prove that, for each $i = 1, \dots, n$, $u_i = v_i x_i$ for some element $v_i \in K[x_1, \dots, \widehat{x_i}, \dots, x_n]^{ev}$. Note that $0 = \delta(0) = \delta(x_i^2) = u_i x_i - x_i u_i = 2u_i x_i$ since u_i is odd, and so $u_i x_i = 0$ since $\frac{1}{2} \in K$. It follows that $u_i = v_i x_i$ for some element $v_i \in K[x_1, \dots, \widehat{x_i}, \dots, x_n]^{ev}$. If $n = 1$ then $v_i \in K$, and so $\delta = \text{sad}(\frac{1}{2}v_i)$, and we are done. So, let $n \geq 2$.

Step 2. We claim that, for each pair $i \neq j$,

$$v_i|_{x_j=0} = v_j|_{x_i=0}. \quad (23)$$

Evaluating the skew derivation δ at the element $0 = x_i x_j + x_j x_i$ and taking into account that all the elements u_i are odd we get

$$\begin{aligned} 0 &= u_i x_j - x_i u_j + u_j x_i - x_j u_i = 2(u_i x_j + u_j x_i) \\ &= 2(v_i x_i x_j + v_j x_j x_i) = 2(v_i - v_j) x_i x_j. \end{aligned}$$

This means that $v_i - v_j \in (x_i, x_j)$ since $\frac{1}{2} \in K$, or, equivalently, $v_i|_{x_i=0, x_j=0} = v_j|_{x_i=0, x_j=0}$. By Step 1, this equality can be written as (23).

Step 3. Note that $\delta(x_1) = v_1 x_1 = \text{sad}(\frac{1}{2}v_1)(x_1)$ since v_1 is even, and so $(\delta - \text{sad}(\frac{1}{2}v_1))(x_1) = 0$. Changing δ for $\delta - \text{sad}(\frac{1}{2}v_1)$ one can assume that $\delta(x_1) = 0$, i.e. $v_1 = 0$. Then, by (23), $v_i|_{x_1=0} = 0$ for all $i = 2, \dots, n$, and so $v_i \in (x_1)$ for all $i = 2, \dots, n$. The idea of the proof is to continue in this way killing the elements v_i . Namely, we are going to prove by induction on k that, for each k such that $1 \leq k \leq n$, by adding to δ a well chosen inner skew derivation we have

$$\delta(x_1) = \dots = \delta(x_k) = 0, \quad \delta(x_i) \in (x_1 \cdots x_k x_i), \quad k < i \leq n. \quad (24)$$

The case $k = 1$ has just been established. Suppose that (24) holds for k , we have to prove the same statement for $k + 1$. Note that $v_{k+1}x_{k+1} = \delta(x_{k+1}) \in (x_1 \cdots x_k x_{k+1})$ (see Steps 1 and 2), hence $v_{k+1} = x_1 \cdots x_k v$ for some $v \in K[x_{k+2}, \dots, x_n]$. Consider the derivation $\delta' := \delta - \text{sad}(\frac{1}{2}v_{k+1})$. For each $i = 1, \dots, k$, $\delta'(x_i) = \delta(x_i) = 0$ as $v_{k+1} \in (x_1 \cdots x_k)$; and

$$\begin{aligned}\delta'(x_{k+1}) &= v_{k+1}x_{k+1} - \frac{1}{2}(v_{k+1}x_{k+1} - (-1)x_{k+1}v_{k+1}) \\ &= v_{k+1}x_{k+1} - \frac{1}{2}(v_{k+1}x_{k+1} + v_{k+1}x_{k+1}) = 0.\end{aligned}$$

So, we have proved the first part of (24) for $k + 1$, namely, that

$$\delta'(x_1) = \cdots = \delta'(x_{k+1}) = 0.$$

So, changing δ for δ' one can assume that

$$\delta(x_1) = \cdots = \delta(x_{k+1}) = 0.$$

These conditions imply that $v_1 = \cdots = v_{k+1} = 0$ (by Step 1). If $n = k + 1$, we are done. So, let $k + 1 < n$. Then, by (23), for each $i > k + 1$, $v_i \in \cap_{j=1}^{k+1} (x_j) = (x_1 \cdots x_{k+1})$, hence $\delta(x_i) = v_i x_i \in (x_1 \cdots x_{k+1} x_i)$, and we are done. By induction, (24) is true for all k . In particular, for $k = n$ one has $\delta = 0$. This means that δ is an inner skew derivation, as required. \square

By Theorem 3.1, any skew K -derivation δ of Λ_n is a unique sum $\delta = \delta^{ev} + \delta^{od}$ of *even* and *odd* skew derivations, and $\delta^{ev} := \frac{1}{2}\text{sad}(a)$ for a *unique* element $a \in \Lambda_n^{ev}$. The next corollary describes explicitly the skew derivations δ^{ev} and δ^{od} .

Corollary 3.2 *Let K be a commutative ring with $\frac{1}{2} \in K$, δ be a skew K -derivation of $\Lambda_n(K)$, and, for each $i = 1, \dots, n$, $\delta(x_i) = u_i^{ev} + u_i^{od}$ for unique elements $u_i^{ev} \in \Lambda_n^{ev}$ and $u_i^{od} \in \Lambda_n^{od}$. Then*

1. $\delta^{od} = \sum_{i=1}^n u_i^{ev} \partial_i$, and
2. $\delta^{ev} = \frac{1}{2}\text{sad}(a)$ where the unique element $a \in \Lambda_n^{ev}$ is given by the formula

$$a = \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(u_{i+1}^{od}) + \partial_1(u_1^{od}).$$

Proof. 1. This statement has been proved already in the proof of Theorem 3.1.

2. For each $i = 1, \dots, n$, on the one hand $\delta^{ev}(x_i) = (\delta - \delta^{od})(x_i) = u_i^{od}$; on the other, $\delta^{ev}(x_i) = \frac{1}{2}\text{sad}(a)(x_i) = \frac{1}{2}2x_i a = x_i a$. So, the element a is a solution to the system of equations

$$\begin{cases} x_1 a = u_1^{od} \\ x_2 a = u_2^{od} \\ \vdots \\ x_n a = u_n^{od}. \end{cases}$$

By Theorem 2.4 and the fact that $a \in \Lambda_n'^{ev}$ (i.e. $a_n = 0$), we have

$$a = \sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(u_{i+1}^{od}) + \partial_1(u_1^{od}). \quad \square$$

Definition. An ideal \mathfrak{a} of Λ_n is called a *skew differential ideal* if $\delta(\mathfrak{a}) \subseteq \mathfrak{a}$ for all $\delta \in \text{SDer}_K(\Lambda_n)$.

Lemma 3.3 *Let K be a commutative ring with $\frac{1}{2} \in K$, $\mathcal{I}(K)$ be the set of ideals of the ring K , $\text{SDI}(\Lambda_n)$ be the set of all skew differential ideals of Λ_n . Then the map*

$$\mathcal{I}(K) \rightarrow \text{SDI}(\Lambda_n), \quad I \mapsto I\Lambda_n,$$

is a bijection. In particular, if K is a field of characteristic $\neq 2$, then Λ_n is a skew differentially simple algebra, i.e. 0 and Λ_n are the only skew differential ideals of Λ_n .

Proof. The map $I \mapsto I\Lambda_n$ is well-defined and injective. It remains to prove that it is surjective. Let \mathfrak{a} be a skew differential ideal of Λ_n . First, let us show that

$$\mathfrak{a} = \bigoplus_{\alpha \in \mathcal{B}_n} (\mathfrak{a} \cap Kx^\alpha),$$

i.e. if $a = \sum_{\alpha \in \mathcal{B}_n} \lambda_\alpha x^\alpha \in \mathfrak{a}$, $\lambda_\alpha \in K$, then all $\lambda_\alpha x^\alpha \in \mathfrak{a}$. The case $a = 0$ is trivial. So, let $a \neq 0$ and $i := \max\{|\alpha| \mid \lambda_\alpha \neq 0\}$. We use induction on i . The case $i = 0$ is obvious. So, let $i > 0$. Then, $\lambda_\alpha = \partial^\alpha(a) \in \mathfrak{a}$ for each α such that $|\alpha| = i$ where $\partial^\alpha := \partial_n^{\alpha_n} \partial_{n-1}^{\alpha_{n-1}} \cdots \partial_1^{\alpha_1}$. Applying induction to the element $a - \sum_{|\alpha|=i} \lambda_\alpha x^\alpha \in \mathfrak{a}$, we get the result. So, $\mathfrak{a} = \bigoplus_{\alpha \in \mathcal{B}_n} \mathfrak{a}_\alpha x^\alpha$ for some ideals \mathfrak{a}_α of K . Let $I := \mathfrak{a}_0$. On the one hand, $I\Lambda_n \subseteq \mathfrak{a}$, and so $I \subseteq \mathfrak{a}_\alpha$ for all $\alpha \in \mathcal{B}_n$. On the other, $\mathfrak{a}_\alpha = \partial^\alpha(\mathfrak{a}_\alpha x^\alpha) \subseteq I$, hence $\mathfrak{a} = I\Lambda_n$. So, the map $I \mapsto I\Lambda_n$ is a surjection. \square

4 Normal elements of the Grassmann algebras

In this section, it is proved that the set of ‘generic’ normal non-units forms no more than two orbits under the action of the group $G := \text{Aut}_K(\Lambda_n)$ (Theorem 4.3). The stabilizers of elements from each orbit are found (Lemma 4.4 and Lemma 4.6).

In this section, K is a *reduced commutative* ring with $\frac{1}{2} \in K$ and $n \geq 2$. Recall that an element r of a ring R is called a *normal* element if $rR = Rr$. Each unit is a normal element.

Recall that $G := \text{Aut}_K(\Lambda_n)$ is the group of K -automorphisms of Λ_n . Consider some of its subgroups:

- $\Omega := \{\omega_{1+a} \mid a \in \Lambda_n'^{od}\}$, where $\omega_u : \Lambda_n \rightarrow \Lambda_n$, $x \mapsto uxu^{-1}$, is an inner automorphism.
- $\Gamma := \{\gamma_b \mid \gamma_b(x_i) = x_i + b_i, b_i \in \Lambda_n'^{od} \cap \mathfrak{m}^3, i = 1, \dots, n\}$, $b = (b_1, \dots, b_n)$,

- $\mathrm{GL}_n(K)^{op} := \{\sigma_A \mid \sigma_A(x_i) = \sum_{j=1}^n a_{ij}x_j, A = (a_{ij}) \in \mathrm{GL}_n(K)\}.$

For each $a \in \Lambda_n^{od}$ and $x \in \Lambda_n$, $\omega_{1+a}(x) = x + [a, x]$ (Lemma 2.8.(3), [1]). Note that $G = (\Omega \rtimes \Gamma) \rtimes \mathrm{GL}_n(K)^{op}$ (Theorem 2.14, [1]). So, each element $\sigma \in G$ has the unique presentation as the product $\sigma = \omega_{1+a}\gamma_b\sigma_A$ where $\omega_{1+a} \in \Omega$ ($a \in \Lambda_n^{od}$), $\gamma_b \in \Gamma$, $\sigma_A \in \mathrm{GL}_n(K)^{op}$ where $\Lambda_n^{od} := \oplus_i \Lambda_{n,i}$ and i runs through all odd natural numbers such that $1 \leq i \leq n-1$. For more information on the group G , the reader is referred to [2] and [4] where K is a field, and to [1] where K is a commutative ring.

Theorem 4.1 [1] *Let K be a reduced commutative ring with $\frac{1}{2} \in K$. Then each element $\sigma \in G$ is a unique product $\sigma = \omega_{1+a}\gamma_b\sigma_A$ where $a \in \Lambda_n^{od}$ and*

1. $\sigma(x) = Ax + \dots$ (i.e. $\sigma(x) \equiv Ax \pmod{\mathfrak{m}}$) for some $A \in \mathrm{GL}_n(K)$,
2. $b = A^{-1}\sigma(x)^{od} - x$, and
3. $a = -\frac{1}{2}\gamma_b(\sum_{i=1}^{n-1} x_1 \cdots x_i \partial_i \cdots \partial_1 \partial_{i+1}(a'_{i+1}) + \partial_1(a'_1))$ where $a'_i := (A^{-1}\gamma_b^{-1}(\sigma(x)^{ev}))_i$, the i 'th component of the column-vector $A^{-1}\gamma_b^{-1}(\sigma(x)^{ev})$.

Remark. In the above theorem the following abbreviations are used

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}, \sigma(x) = \begin{pmatrix} \sigma(x_1) \\ \vdots \\ \sigma(x_n) \end{pmatrix}, \sigma(x)^{ev} = \begin{pmatrix} \sigma(x_1)^{ev} \\ \vdots \\ \sigma(x_n)^{ev} \end{pmatrix}, \sigma(x)^{od} = \begin{pmatrix} \sigma(x_1)^{od} \\ \vdots \\ \sigma(x_n)^{od} \end{pmatrix},$$

any element $u \in \Lambda_n$ is a unique sum $u = u^{ev} + u^{od}$ of its even and odd components. Note that the inversion formula for γ_b^{-1} is given in [1].

Let \mathcal{N} be the set of all normal elements of the Grassmann algebra $\Lambda_n = \Lambda_n(K)$ and let \mathcal{U} be the set of all units of Λ_n . Then $\mathcal{U} \subseteq \mathcal{N}$. By (15), the set \mathcal{N} is a disjoint union of its G -invariant subsets,

$$\mathcal{N} = \cup_{i=0}^n \mathcal{N}_i, \quad \mathcal{N}_i := \{a \in \mathcal{N} \mid a = a_i + \dots, 0 \neq a_i \in \Lambda_{n,i}\}.$$

Clearly, $\mathcal{N}_0 = \mathcal{U}$. Similarly, by (15), the set \mathcal{U} is a disjoint union of its G -invariant subsets \mathcal{U}_i ,

$$\mathcal{U} = \cup_{i=0}^n \mathcal{U}_i, \quad \mathcal{U}_i := \{a \in \mathcal{U} \mid a = a_0 + a_i + \dots, a_0 \in K^*, 0 \neq a_i \in \Lambda_{n,i}\}.$$

Lemma 4.2 $\Lambda_n^{ev} \cup \Lambda_n^{od} \subseteq \mathcal{N}$.

Proof. It is obvious. \square

The next result shows that ‘generic’ normal non-unit elements of Λ_n (i.e. the set \mathcal{N}_1) form a single G -orbit if n is even, and two G -orbits if n is odd.

Theorem 4.3 *Let K be a field of characteristic $\neq 2$ and $\Lambda_n = \Lambda_n(K)$. Then*

1. $\mathcal{N}_1 = Gx_1$ if n is even.

2. $\mathcal{N}_1 = Gx_1 \cup G(x_1 + x_2 \cdots x_n)$ is the disjoint union of two orbits if n is odd.

Proof. The elements x_1 and $y := x_1 + x_2 \cdots x_n$ are normal. First, let us prove that if n is odd then the orbits Gx_1 and Gy are distinct. Suppose that they coincide, i.e. $y = \sigma(x_1)$ for some automorphism $\sigma \in G$, we seek a contradiction. By Theorem 4.1, $\sigma = \omega_{1+a}\gamma_b\sigma_A$. By taking the equality $\sigma(x_1) = y$ modulo the ideal \mathfrak{m}^2 , we have $\sigma_A(x_1) = x_1$, hence

$$x_1 + x_2 \cdots x_n = y = \sigma(x_1) = \omega_{1+a}\gamma_b(x_1) = \omega_{1+a}(x_1 + b_1) = x_1 + b_1 + [a, x_1 + b_1].$$

Equating the odd parts of both ends of the equalities above gives $x_1 = x_1 + b_1$, hence $b_1 = 0$ and $x_2 \cdots x_n = [a, x_1] \in (x_1)$, a contradiction. Therefore, the orbits Gx_1 and Gy are distinct.

It remains to prove that $\mathcal{N}_1 \subseteq Gx_1$ and $\mathcal{N}_1 \subseteq Gx_1 \cup Gy$ in the first and the second case respectively.

Let $a = a_1 + a_2 + \cdots \in \mathcal{N}_1$ where all $a_i \in \Lambda_{n,i}$ and $0 \neq a_1 \in \Lambda_{n,1}$. Up to the action of the group $\mathrm{GL}_n(K)^{op}$, one can assume that $a_1 = x_1$. The automorphism $\gamma : x_1 \mapsto x_1 + a_3 + a_5 + \cdots$, $x_i \mapsto x_i$, $i \geq 2$, is an element of the group Γ . Now, $a = \gamma(x_1) + a^{ev}$ where $a^{ev} := a_2 + a_4 + \cdots$ is the even part of the element a , and so $\gamma^{-1}(a) = x_1 + \gamma^{-1}(a^{ev})$. Note that $\gamma^{-1}(a^{ev})$ is an even element of the set $\Lambda_n^{ev} \cap \mathfrak{m}^2$. Therefore, up to the action of the group Γ , one can assume that $a = x_1 + a^{ev}$ where a^{ev} is an even element of \mathfrak{m}^2 . Since $\frac{1}{2} \in K$, the element a^{ev} is the unique sum $a^{ev} = 2\alpha x_1 + \beta$ where α and β are respectively odd and even elements of the Grassmann algebra $K[x_2, \dots, x_n]$ and $\alpha, \beta \in \mathfrak{m}$. Applying the inner automorphism $\omega_{1-\alpha} = (\omega_{1+\alpha})^{-1}$ to the equality

$$a = x_1 + 2\alpha x_1 + \beta = x_1 + [\alpha, x_1] + \beta = \omega_{1+\alpha}(x_1) + \beta = \omega_{1+\alpha}(x_1 + \beta)$$

we have the equality $\omega_{1-\alpha}(a) = x_1 + \beta$. So, up to the action of the group Ω , one can assume that $a^{ev} = \beta \in K[x_2, \dots, x_n]_{\geq 2}^{ev}$. If $\beta = 0$ then we are done. So, let $\beta \neq 0$.

Case 1. $\beta x_i = 0$ for all $i = 2, \dots, n$, i.e. $\beta = \lambda x_2 \cdots x_n$, $\lambda \in K$, hence n must be odd since β is even. In this case, $a = \sigma_A(y)$ where $\sigma_A(x_2) := \lambda^{-1}x_2$ and $\sigma_A(x_i) = x_i$ for all $i \neq 2$, and we are done.

Case 2. $\beta x_i \neq 0$ for some $i \geq 2$. We aim to show that this case is impossible, we seek a contradiction. Let $\beta = a_{2m} + \cdots$ where $a_{2m} \in K[x_2, \dots, x_n]_{2m}$, $a_{2m}x_i \neq 0$, and the three dots mean higher terms with respect to the \mathbb{Z} -grading of the Grassmann algebra Λ_n . The element $a = x_1 + a_{2m} + \cdots$ is normal, and so $ax_i = ba$ for some element $b = b_0 + b_1 + \cdots \in \Lambda_n$ where $b_i \in \Lambda_{n,i}$. In more detail,

$$(x_1 + a_{2m} + \cdots)x_i = (b_0 + b_1 + \cdots)(x_1 + a_{2m} + \cdots).$$

Clearly, $b_0 = 0$. Equating the homogeneous components of degrees $1, \dots, 2m+1$ of both sides of the equality we have the system of equations:

$$\begin{cases} b_1x_1 = x_1x_i, \\ b_2x_1 = 0, \\ \vdots \\ b_{2m-1}x_1 = 0, \\ b_1a_{2m} + b_{2m}x_1 = a_{2m}x_i. \end{cases}$$

The first equation gives $b_1 = -x_i + \lambda x_1$ for some $\lambda \in K$. By taking the last equation modulo the ideal (x_1) of Λ_n we have the following equality in the Grassmann algebra $K[x_2, \dots, x_n]$: $-x_i a_{2m} = a_{2m} x_i$, hence $2a_{2m} x_i = 0$. Dividing by 2, we have the equality $a_{2m} x_i = 0$, which contradicts to the assumption that $a_{2m} x_i \neq 0$. \square

Let $\text{Stab}(x_1) := \{\sigma \in G \mid \sigma(x_1) = x_1\}$ be the *stabilizer* of the element x_1 in G and \mathcal{N}_1 be the set of normal elements as in Theorem 4.3. By Theorem 4.3.(1), the map

$$G/\text{Stab}(x_1) \rightarrow \mathcal{N}_1, \quad \sigma \text{Stab}(x_1) \mapsto \sigma(x_1), \quad (25)$$

is a bijection where K is a field of characteristic $\neq 2$.

The next lemma describes the stabilizer $\text{Stab}(x_1)$.

Lemma 4.4 *Let K be a reduced commutative ring with $\frac{1}{2} \in K$, $\Omega_{x_1} := \Omega \cap \text{Stab}(x_1) = \{\omega_{1+a} \mid a \in (x_1) \cap \Lambda_n^{od}\}$, $\Gamma_{x_1} := \Gamma \cap \text{Stab}(x_1) = \{\gamma_b \mid b = (0, b_2, \dots, b_n) \in \Lambda_{n, \geq 3}^{od}\}$, and $\text{GL}_n(K)_{x_1}^{op} := \text{GL}_n(K)^{op} \cap \text{Stab}(x_1) = \{\sigma_A \mid \sigma_A(x_1) = x_1, A \in \text{GL}_n(K)\}$. Then*

$$\text{Stab}(x_1) = \Omega_{x_1} \Gamma_{x_1} \text{GL}_n(K)_{x_1}^{op} = (\Omega_{x_1} \rtimes \Gamma_{x_1}) \rtimes \text{GL}_n(K)_{x_1}^{op}.$$

Proof. The last equality follows from the equality $G = (\Omega \rtimes \Gamma) \rtimes \text{GL}_n(K)^{op}$ (Theorem 2.14, [1]) provided the equality before is true. Let $\mathcal{M} := \Omega_{x_1} \Gamma_{x_1} \text{GL}_n(K)_{x_1}^{op}$. Then $\text{Stab}(x_1) \supseteq \mathcal{M}$. It remains to prove the reverse inclusion. Let $\sigma \in \text{Stab}(x_1)$. By Theorem 4.1, $\sigma = \omega_{1+a} \gamma_b \sigma_A$. Since $\sigma(x_1) \equiv \sigma_A(x_1) \pmod{\mathfrak{m}^2}$ and $\sigma(x_1) = x_1$ we must have $\sigma_A \in \text{GL}_n(K)_{x_1}^{op}$. Now,

$$x_1 = \sigma(x_1) = \omega_{1+a}(x_1 + b_1) = x_1 + b_1 + [a, x_1 + b_1] = x_1 + b_1 + 2a(x_1 + b_1),$$

and equating the odd parts of the elements at both ends of the equalities we see that $x_1 = x_1 + b_1$, hence $b_1 = 0$, and so $\gamma_b \in \Gamma_{x_1}$. Putting $b_1 = 0$ in the equalities above gives $x_1 = x_1 + 2ax_1$, hence $0 = x_1^{ev} = (x_1 + 2ax_1)^{ev} = 2ax_1$, and so $a \in (x_1)$. This means that $\sigma \in \Omega_{x_1}$, as required. \square

Corollary 4.5 *Let K be a reduced commutative ring with $\frac{1}{2} \in K$, and I be a non-empty subset of $\{1, \dots, n\}$. Then*

$$\cap_{i \in I} \text{Stab}(x_i) = (\cap_{i \in I} \Omega_{x_i}) \cdot (\cap_{i \in I} \Gamma_{x_i}) \cdot (\cap_{i \in I} \text{GL}_n(K)_{x_i}^{op}) = (\cap_{i \in I} \Omega_{x_i}) \rtimes (\cap_{i \in I} \Gamma_{x_i}) \rtimes (\cap_{i \in I} \text{GL}_n(K)_{x_i}^{op}).$$

Proof. This follows from Lemma 4.4 (and uniqueness of the decomposition $\sigma = \omega_{1+a} \gamma_b \sigma_A$). \square

If K is a field of characteristic $\neq 2$ and n is an odd number then, by Theorem 4.3.(2), the map (where $y := x_1 + x_2 \cdots x_n$)

$$G/\text{Stab}(x_1) \cup G/\text{Stab}(y) \rightarrow \mathcal{N}, \quad \sigma \text{Stab}(x_1) \mapsto \sigma(x_1), \quad \tau \text{Stab}(y) \mapsto \tau(y), \quad (26)$$

is a bijection. The next lemma describes the stabilizer $\text{Stab}(x_1 + x_2 \cdots x_n)$.

Lemma 4.6 *Let K be a reduced commutative ring with $\frac{1}{2} \in K$, and $n \geq 3$ be an odd number. Then $\text{Stab}(x_1 + x_2 \cdots x_n) = \{\omega_{1+\frac{1}{2}\partial_1\gamma_b\sigma_A(x_2\cdots x_n)+x_1c}\gamma_b\sigma_A \mid \gamma_b \in \Gamma_{x_1}, \sigma_A \in \text{GL}_n(K)_{x_1}^{op}, (1 - x_1\partial_1)\gamma_b\sigma_A(x_2\cdots x_n) = x_2\cdots x_n, c \in K[x_2, \dots, x_n]^{ev}\}$.*

Proof. Let $y := x_1 + x_2 \cdots x_n$ and $\sigma \in \text{Stab}(y)$. Note that $\sigma = \omega_{1+a}\gamma_b\sigma_A$. Since $x_1 \equiv y \equiv \sigma(y) \equiv \sigma(x_1) \pmod{\mathfrak{m}^2}$, we must have $\sigma_A \in \text{GL}_n(K)_{x_1}^{op}$. Then

$$\begin{aligned} x_1 + x_2 \cdots x_n &= y = \sigma(y) = \omega_{1+a}\gamma_b(x_1 + \sigma_A(x_2 \cdots x_n)) = \omega_{1+a}(x_1 + b_1 + \gamma_b\sigma_A(x_2 \cdots x_n)) \\ &= x_1 + b_1 + \gamma_b\sigma_A(x_2 \cdots x_n) + [a, x_1 + b_1]. \end{aligned}$$

Equating the odd parts of the beginning and the end of the series of equalities above we obtain $x_1 = x_1 + b_1$, hence $b_1 = 0$, i.e. $\gamma_b \in \Gamma_{x_1}$, and then

$$\gamma_b\sigma_A(x_2 \cdots x_n) - x_2 \cdots x_n = 2x_1a. \quad (27)$$

Each element $u \in \Lambda_n$ is a unique sum $u = x_1\alpha + \beta$ for unique elements $\alpha, \beta \in K[x_2, \dots, x_n]$. Clearly, $\alpha = \partial_1(u)$ and $\beta = (1 - x_1\partial_1)(u)$. The odd element a is a unique sum $a = x_1c + d$ for some elements $c \in K[x_2, \dots, x_n]^{ev}$ and $d \in K[x_2, \dots, x_n]_{\geq 1}^{od}$. The equation (27) can be written as follows $\gamma_b\sigma_A(x_2 \cdots x_n) - x_2 \cdots x_n = 2x_1d$. This equality is equivalent to two equalities $d = \frac{1}{2}\partial_1(\gamma_b\sigma_A(x_2 \cdots x_n) - x_2 \cdots x_n) = \frac{1}{2}\partial_1\gamma_b\sigma_A(x_2 \cdots x_n)$ and $0 = (1 - x_1\partial_1)(\gamma_b\sigma_A(x_2 \cdots x_n) - x_2 \cdots x_n) = (1 - x_1\partial_1)\gamma_b\sigma_A(x_2 \cdots x_n) - x_2 \cdots x_n$. This finishes the proof of the lemma. \square

Let \mathcal{U}'_1 be the image of the injection $K^* \times \mathcal{N}_1 \rightarrow \mathcal{U}_1$, $(\lambda, u) \mapsto \lambda(1 + u)$. The next corollary follows from Theorem 4.3.

Corollary 4.7 *Let K be a field of characteristic $\neq 2$, and $\Lambda_n = \Lambda_n(K)$. Then*

1. $\mathcal{U}'_1 = \cup_{\lambda \in K^*} G \cdot \lambda(1 + x_1)$ is the disjoint union of orbits if n is even.
2. $\mathcal{U}'_1 = \cup_{\lambda \in K^*} (G \cdot \lambda(1 + x_1) \cup G \cdot \lambda(1 + x_1 + x_2 \cdots x_n))$ is the disjoint union of orbits if n is odd.

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